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On the Modified Complementary Mukherjee-Islam Distribution

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Abstract

This study considered a modified version of the Mukherjee-Islam distribution referred to as the Modified Complementary Mukherjee-Islam. The new distribution was obtained using the new quasi-transmutation technique. The study also derived some essential properties of the modified distribution based on the baseline Mukherjee-Islam distribution to illustrate its inherent flexibility.

Keywords: *Quasi-transmuted complementary Mukherjee-Islam distribution, Moment generating function, Cumulant generating function, Hazard function, Order statistics*

1.0 INTRODUCTION

Obtaining generalized distributions from existing distributions has been an important research interest of applied statisticians (Jamal *et. al*, 2017). They noted further that existing distributions are generally limited in their functional forms and potentials of flexibility in them are often restricted. The need for generalized distributions is captured by Subramanian and Rather (2018) who opined that existing classical distributions may not fit well the data under consideration. Expanding the inherent flexibility in basic classical density functions is an old practice which modern day researchers in applied statistics has kept faith with. This is often achieved by the introduction of one or more extra parameters into a baseline distribution of interest using different techniques. This is in line with Eghwerido *et. al*. (2019)'s opinion that a new distribution could be generated by compounding two or more distributions.

Many generalized distributions exist in literature. Some of these include the Kumaraswamy generated family of distributions by Cordeiro and Castro (2009); Beta Normal distribution by Eugene *et. al*. (2002); Beta Gumbel distribution by Nadarajah and Kotz (20004); Beta Frechet distribution by Nadarajah and Gupta (20004); Marshall-Olkin Extended Burr Type XII distribution by Al-Saiari *et. al.* (2014); Marshall-Olkin Extended Burr Type III distribution by Al-Saiari *et. al.* (2016); Weibull Frechet distribution by Afify *et al.* (2016); Generalized Inverted exponential distribution by Abouammoh and Alshingiti (2009); Transmuted Inverse Exponential distribution by Oguntunde and Adejumo (2014); Extended Generalized distribution by Olapade (2014); Harris Extended distribution by Pinho et al. (2015); Exponentiated Generalized Extended Exponential distribution by Thiago et al. (2016); Extended Generalized Exponential distribution by Dey *et. al.* (2017); and Odd Generalized Exponential Power Function distribution by Hassan et al. (2019).

The baseline distribution selected for this study is the Complementary Murhkerjee-Islam distribution defined below

$$f(x) = bc(xc)^{b-1}, 0 \le x \le \frac{1}{c}; b, c \ge 1 \qquad \ldots\ldots(1.0)$$

For (1.0), the cumulative distribution function (c. d. f) is given as

$$F(x) = (xc)^{b}, 0 \le x \le \frac{1}{c}; b, c \ge 1 \qquad \ldots\ldots(2.0)$$

The quasi-transmutation technique is derived from the general quadratic equation

$$Ax^2 + Bx + C = 0 \qquad \ldots\ldots(3.0)$$

Thus, a random variable X is said to have a Quasi-Transmuted distribution if its cdf is given as

$$G^{*}_{Q-T}(x) = a[F(x)]^2 + (1-a)F(x), a > 0 \qquad \ldots\ldots(4.0)$$

where F(x) is the cdf of the selected baseline distribution

The probability density function (p. d. f.) of the Quasi-Transmuted distribution is obtained from (4.0) using (5.0). That is,

$$g^{*}_{Q-T}(x) = [G^{*}_{Q-T}(x)]' = 2af(x)F(x) + (1-a)f(x), a > 0 \qquad \ldots\ldots(5.0)$$

It should be noted that $G^{*}_{Q-T}(x)$ is a valid cdf since $G^{*}_{Q-T}(-\infty) = 0$ and $G^{*}_{Q-T}(\infty) = 1$. The implication of this is that $g^{*}_{Q-T}(x)$ is also a valid pdf.

2.0 THE QUASI-TRANSMUTED COMPLEMENTARY MUKHERJEE-ISLAM DISTRIBUTION

The Quasi-transmuted complementary Mukherjee-Islam distribution is a continuous density function with cdf and pdf defined as

$$G^{*}_{Q-T}(x) = a\left[(xc)^{b}\right]^{2} + \left[(1\text{-}a)(xc)^{b}\right],\ 0 \le x \le \frac{1}{c};\ b, c \ge 1;\ a\ \varepsilon\ (0, \frac{1}{2}, 1) \qquad \ldots\ldots(6.0)$$

$$g^{*}_{Q-T}(x) = 2abc(xc)^{2b-1} + (1\text{-}a)bc(xc)^{b-1},\ 0 \le x \le \frac{1}{c};\ b, c \ge 1; a\ \varepsilon\ (0, \frac{1}{2}, 1) \qquad \ldots\ldots(7.0)$$

Some essential properties of the Quasi-Transmuted Complementary Mukherjee-Islam distribution shall be considered next.

2.1 CHARACTERIZATION OF THE QUASI-TRANSMUTED COMPLEMENTARY MUKHERJEE-ISLAM DISTRIBUTION

2.1.1 MOMENTS

The k^{th} moment about zero for a random variable X from the Quasi-Transmuted Complementary Mukherjee-Islam distribution is

$$\mu_k^1 = E(X^K) = \int_0^{\frac{1}{c}} x^k \, g_{Q-T}^*(x)\,dx$$

$$= \int_0^{\frac{1}{c}} x^k \left[2abc(xc)^{2b-1} + (1-a)bc(xc)^{b-1}\right]dx$$

$$= \int_0^{\frac{1}{c}} \left[2abc(xc)^{2b-1}(x^k) + (1-a)bc(xc)^{b-1}(x^k)\right]dx; b, c \geq 1;\ a\ \varepsilon\ (0, \frac{1}{2} \& 1)$$

$$= \frac{2ab}{2b+k}c^{-k} + \frac{b}{b+k}c^{-k} - \frac{ab}{b+k}c^{-k}$$

$$= \left[\frac{2ab}{2b+k} + \frac{b}{b+k} - \frac{ab}{b+k}\right]c^{-k}; b, c \geq 1;\ a\ \varepsilon\ (0, \frac{1}{2}, 1)$$

$$\therefore\ \mu_1^1 = \mathrm{E}(X) = mean = \frac{b}{b+1}, \frac{4b^2+3bk}{2(b+k)(2b+k)}, \frac{2b}{2b+k} \text{ for a = 0, } \frac{1}{2} \text{ and } 1 \text{ respectively when c=1}$$

The kth moment about the mean for a random variable X from the Quasi-Transmuted Complementary Mukherjee-Islam distribution is defined as

$$\mu_k = \mathrm{E}(X-\mu)^k$$

$$= \int_0^{\frac{1}{c}} (x-\mu)^k g_{Q-T}^*(x)dx = \int_0^{\frac{1}{c}} (x-\mu)^k \left[2abc(xc)^{2b-1}x^k + (1-a)bc(xc)^{b-1}x^k\right]dx$$

Specifically, when k = 2, $\mu_k = \mu_2$, *the variance*, σ_X^2. Hence,

$$Variance\,(X) = \sigma_X^2 = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

$$= \frac{b}{(b+2)(b+1)^2}, \frac{-8b^4+68b^2+96b}{[2(b+2)(2b+2)]^2} \text{ and } \frac{4b}{(2b+2)^2} \text{ for } a = 0, \frac{1}{2} \text{ and } 1 \text{ respectively when } c=1$$

2.1.2 MOMENT GENERATING FUNCTION

The moment generating function (m.g.f.) of a random variable X from the Quasi-Transmuted Complementary Mukherjee-Islam distribution is defined as

$$M_X(t) = \mathrm{E}(e^{tX}) = \int_0^{\frac{1}{c}} e^{tX} g_{Q-T}^*(x)dx$$

$$= \int_0^{\frac{1}{c}} \left[1 + tX + \frac{(tX)^2}{2!} + \ldots\right] g_{Q-T}^*(x)dx$$

$$= \int_0^{\frac{1}{c}} \sum_{j=0}^{\infty} \frac{t^j}{j!} X^j g_{Q-T}^*(x)dx$$

$$=\sum_{j=0}^{\infty}\frac{t^j}{j!}\mu_j^1$$

$$=\sum_{j=0}^{\infty}\frac{t^j}{j!}\left[\frac{2ab}{2b+j}+\frac{b}{b+j}-\frac{ab}{b+j}\right]c^{-j};b,c\geq 1;\ a\ \varepsilon\ (0,\frac{1}{2},1)$$

2.1.3 CHARACTERISTIC FUNCTION

The characteristic function (c. f.) of a random variable X from the Quasi-Transmuted Complementary Mukherjee-Islam distribution is defined as

$$\phi_x(t)=M_X(it)$$

$$=\mathrm{E}(e^{itX})$$

$$=\int_0^{\frac{1}{c}}e^{itX}g_{Q-T}^{*}(x)dx$$

$$=\sum_{j=0}^{\infty}\frac{(it)^j}{j!}\mu_j'$$

$$=\sum_{j=0}^{\infty}\frac{(it)^j}{j!}\left[\frac{2ab}{2b+j}+\frac{b}{b+j}-\frac{ab}{b+j}\right]c^{-j};b,c\geq 1;a\ \varepsilon\ (0,\frac{1}{2},1)$$

2.1.4 CUMULANT GENERATING FUNCTION

The cumulant generating function (c. g. f.) of a random variable X from the Quasi-Transmuted Complementary Mukherjee-Islam distribution is defined as

$$K_X(t)=In\left[M_X(t)\right]$$

$$=In\left[\sum_{j=o}^{\infty}\frac{t^j}{j!}\mu_j^1\right]$$

$$=In\left[\sum_{j=0}^{\infty}\frac{t^j}{j!}[\frac{2ab}{2b+j}+\frac{b}{b+j}-\frac{ab}{b+j}]c^{-j}\right],b,c\geq 1;a\ \varepsilon\ (0,\frac{1}{2},1)$$

2.1.5 HAZARD FUNCTION

The hazard function (h. f.) of a random variable X from the Quasi-Transmuted Complementary Mukherjee-Islam distribution is defined as

$$h(x) = \frac{g^*_{Q-T}(x)}{1-G^*_{Q-T}(x)}$$

$$= \frac{2abc(xc)^{2b-1}+(1-a)bc(xc)^{b-1}}{1-a\left[(xc)^b\right]^2+(1-a)(xc)^b}, 0 \le x \le \frac{1}{c}; b,c \ge 1;\ a\ \varepsilon\ (0,\frac{1}{2},1)$$

2.1.6 REVERSE HAZARD FUNCTION

The reserve hazard function (r. h. f.) of a random variable X from the Quasi-Transmuted Complementary Mukherjee-Islam distribution is defined as

$$h_r(x) = \frac{g^*_{Q-T}(x)}{G^*_{Q-T}(x)}$$

$$= \frac{2abc(xc)^{2b-1}+(1-a)bc(xc)^{b-1}}{a\left[(xc)^b\right]^2+(1-a)(xc)^b}, 0 \le x \le \frac{1}{c}; b,c \ge 1;\ a\ \varepsilon\ (0,\frac{1}{2},1)$$

2.1.7 SURVIVAL FUNCTION

The survival function (s.f.) of a random variable X from the Quasi-Transmuted Complementary Mukherjee-Islam distribution is defined as

$$S_X(x) = 1 - G^*_{Q-T}(x)$$

$$=1-a\left[(xc)^b\right]^2+(1-a)(xc)^b, 0 \le x \le \frac{1}{c}; b,c \ge 1; a \in (0,\frac{1}{2},1)$$

2.1.8 ORDER STATISTICS

Suppose $X_{(1)}, X_{(2)},\ldots, X_{(n)}$ are the order statistics from the random sample $X_1, X_2, \ldots, X_n$ from the Quasi-Transmuted Complementary Mukherjee-Islam distribution with $g^*_{Q-T}(x)$ and $G^*_{Q-T}(x)$ as the pdf and cdf, the rth order statistic where $1 \le r \le n$ is given as

$$h_{(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\, g^*_{Q-T}(x)\left[G^*_{Q-T}(x)\right]^{r-1}\left[1-G^*_{Q-T}(x)\right]^{n-r}$$

$$= \frac{n\,!}{(r-1)!(n-r)!}\left[2abc(xc)^{2b-1}+(1-a)bc(xc)^{b-1}\right]\left[A_1\right]^{r-1}\left[A_2\right]^{n-r}, 0\le x\le \frac{1}{c}; b,c\ge 1; a\in(0,\frac{1}{2},1)$$

where

$A_1 = a\left[(xc)^b\right]^2+(1-a)(xc)^b, 0\le x\le\frac{1}{c}; b,c\ge1; a\in(0,\frac{1}{2},1)$ and

$A_2 = 1-a\left[(xc)^b\right]^2+(1-a)(xc)^b, 0\le x\le\frac{1}{c}; b,c\ge1; a\in(0,\frac{1}{2},1)$

By setting r = n and r =1 in the function ($h_{(r)}(x)$) above, we have the distributions for the largest and lowest order statistics. For the largest order statistic, we have that

$$h_{(r=n)}(x) = \frac{n!}{(n-1)!}\left[A_1\right]^{n-1} g^*_{Q-T}(x)$$

$$= n\,[a\left[(xc)^b\right]^2+(1-a)(xc)^b]^{n-1}[2abc(xc)^{2b-1}+(1-a)bc(xc)^{b-1}], 0\le x\le\frac{1}{c}; b,c\ge1; a\in(0,\frac{1}{2},1)$$

For the lowest order statistic, we have that

$$h_{(r=1)}(x) = n\left[A_2\right]^{n-1} g^*_{Q-T}(x)$$

$$= n[1-a\left[(xc)^b\right]^2+(1-a)(xc)^b]^{n-1}(2abc(xc)^{2b-1}+(1-a)bc(xc)^{b-1}), 0\le x\le\frac{1}{c}; b,c\ge1; a\;\varepsilon\;(0,\frac{1}{2},1)$$

2.1.9 RANDOM NUMBER GENERATION

By the quantile function (i.e the inverse of the cdf), random numbers can be generated for the Quasi-Transmuted Complementary Mukherjee-Islam distribution as follows

Let $a[(xc)^b]^2 + (1-a)(xc)^b = u$ for a ε $(0, \frac{1}{2}, 1)$ where U is a random variable from the Uniform (0, 1) distribution. The random number generation will be done using the possible values of a.

For a=0, $a[(xc)^b]^2 + (1-a)(xc)^b = u$ becomes $(xc)^b = u$ so that $(xc)^b = In\,u$. This implies that $bIn\,(xc) = In\,u \Rightarrow In\,(xc) = \frac{1}{b}In\,u \Rightarrow In\,x + In\,c = \frac{1}{b}In\,u \Rightarrow$

$In\ x = \frac{1}{b} In\ u - In\ c$ so that $x = e^{\frac{1}{b} In\ u - In\ c}$.Thus x may be determined from estimates of b, c and u.

For a=$\frac{1}{2}$, $a[(xc)^b]^2 + (1-a)(xc)^b = u$ becomes$(xc)^{2b} + (xc)^b = 2u$. By choosing b=1,

$(xc)^{2b} + (xc)^b = 2u$ becomes $x^2 + \frac{x}{c} - \frac{2u}{c^2} = 0 \Rightarrow x = \frac{\frac{-1}{c} \pm \sqrt{\frac{1}{c^2} + 4(\frac{2u}{c^2})}}{2}$. Again, x may be determined from estimates of c and u.

For a = 1, $a[(xc)^b]^2 + (1-a)(xc)^b = u$ becomes $(xc)^{2b} = u$ so that $In\ (xc)^{2b} = In\ u \Rightarrow$

$2bIn\ (xc) = In\ u \quad \Rightarrow In\ (xc) = \frac{1}{2b} In\ u \quad \Rightarrow In\ x = \frac{1}{2b} In\ u - In\ c$ so that $x = e^{\frac{1}{2b} In\ u - In\ c}$. Furthermore, x may be determined from estimates of b, c and u.

3.0 CONCLUSION

The Quasi-Transmuted Complementary Mukherjee-Islam distribution has been derived successfully in this study as well as its essential properties. The variance is expected to reduce as the values of b increase; as is expected of any generalized distribution derived from a selected baseline distribution. The method of quasi-transmutation has been shown to have potentials for generating other distributions.

4.0 RECOMMENDATION

The quasi-transmutation technique is therefore recommended for future generalized family of distributions. This is because the technique is very easy to understand since it is defined on the general quadratic equation. Additionally, the technique proposed in the study offers *apriori* determination of the additional parameter.

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